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Thermal-stress concentration near inclusions in viscoelastic random composites

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Abstract The determination of thermal-stress concentrations near inclusions in viscoelastic random composites is concerned with the prediction of the overall response of random nonlinear viscoelastic multi-component media. The continuum considered here is assumed to be subjected to a finite deformation. First Piola's stress tensor and deformation gradient are used as conjugate field variables in a fixed reference state. A nonlinear problem is investigated in a second-order approximation theory when the gradient deformation terms higher than second order are neglected. A convex potential function in a thermo-elastic problem and time functionals in a viscoelastic one are used to construct overall constitutive relations. The technique of surface operators developed by R. Hill and others is used to determine stress concentrations near inclusions for nonlinear matrix creep.

Keywords Concentration · Creep · Stress · Thermo-elasticity

1 Introduction

The determination of stress concentrations near inclusions and the prediction of the overall properties of inhomogeneous media are problems of great importance. This work is concerned with estimating the overall response and local stress distribution in a random multi-component composite with nonlinear thermo-viscoelastic constituents. The thermo-elastic properties of inhomogeneous structures have been studied intensively during the last decades [1–3], etc. Many questions arise in the case of nonlinear thermo-viscoelastic response [4,5]. Composite materials are often used in structural applications due to their well-known advanced properties. The prediction of the behavior of these materials is an important step in the process of its implementation in structural design. Inclusion-reinforced thermo-viscoelastic materials are the subject of this investigation. Much effort has been devoted to the determination of the effective linear thermo-elastic properties, singular stress fields and deformation near the tip of cracks etc. It is usually assumed that the inclusions are imbedded in a linear defect-free continuum. As a result, three topics are important, namely macroscopic interacting cracks and multi-component inclusions [6], distributed microscopic damage [7,8] and third the nonlinear properties of the constituents [5,9,10]. The early analytical work regarding damage of composites used linear elastic fracture mechanics, making it less successful in applications than those applied to metals. The new approach has been created recently which would be a fruitful tool in composite

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micro-mechanics. One of these developed in the last few years is a mathematical model involving a multi-particle effective field method (MEFM) that has reached a level at which many practical and significant problems can be solved [6]. The effective field method is applied to the calculation of the overall dielectric permittivities of composite materials [11] consisting of a homogeneous matrix and a set of spherical inclusions. The main hypothesis of the classical version is the assumption that the field acting on every inclusion in the composite is constant, the same being true for all the inclusions. The predictions of this version of the method are usually in agreement with experimental data and numerical solutions for effective properties of composites if the volume concentration is rather small. So the new version of the effective field method has been developed for the improvement of the predictions of the overall properties of the composites in regions where the volume concentrations of the inclusions is high. This paper is devoted to a study of these models.

2 Problem statement and definition

Increasing the strength and reliability of constructions made from composite materials is largely a complex multi-parametric problem [12,13]. One of its solutions concerns the evaluation of the stress concentration in microstructural elements and the formulation of required durability criteria corresponding to classical methods of strength theory. The service life of products involving composite materials is dependent on the average or maximum cyclic stress both in the matrix and the inclusions, on the number of load cycles etc. Moreover, many behavioral singularities of non-homogenous materials can be given only in terms of nonlinear mechanics [5,14–16]. Therefore, developing algorithms for designing new multi-constituent composites with advanced properties requires an in-depth study of the stress in microstructural elements and the calculation of the overall thermo-elastic parameters within the scope of nonlinear continuum theory. The results given in [1,3,17] involve effective thermo-elastic modules of nonlinear compressible and incompressible random composites containing two components: a matrix and a set of inclusions. The problem addressed in [10] concerns the microstructure stress determination in nonlinear incompressible multi-constituent composites. The present study continues the previous investigations [18–20] and summarizes the problems for compressible materials. The ideas regarding multi-particle effective field methods [11,21] and Mori–Tanaka's scheme [22] are implemented in a new approach. The solution of the first iteration for small inclusion concentration is based on results obtained earlier in [20]. First, a representative volume v_R of the composite body B taken in a reference configuration is considered. It is assumed that the composite specimen is subjected to some non-random system of loading. The stress and strain fields vary from point to point. If every detail of the geometry of a composite were known, overall properties could be calculated exactly and, as a result, local stress concentration as well. In practice, however, except in cases such as those that display periodicity etc., a complete solution could not even be computed. So random media may be used as a useful model useful for the solution of engineering problems. A random medium is understood here as belonging to a family, any member of which may be characterized by a label α that belongs to a sample space A. For a multi-phase composite material it is convenient to introduce the indicator function $f_r(\mathbf{x})$, that takes the value 1 if it lies in phase r and zero otherwise. It depends on α which denotes individual members of a sample space A, defined by $P_r(\mathbf{x})$. Let the probability density of α in space A be $p(\alpha)$. Then the mean value or ensemble average $E[f_r(\mathbf{x})]$ of the indicator function $f_r(\mathbf{x})$ defines the probability of finding phase r at $\mathbf{x} \in B$. Thus

$$P_r(\mathbf{x}) = \mathbf{E}\left[\mathbf{f_r}(\mathbf{x})\right] = \int_{\mathbf{A}} \mathbf{f_r}(\mathbf{x}, \alpha)\mathbf{p}(\alpha)\mathrm{d}\alpha, \tag{1}$$

i.e., the operation of statistical averaging is denoted by the mathematical expectation symbol $E(\cdot)$ which expresses the condition of which set a point belongs to. Likewise the probability $P_{rs}(\mathbf{x}, \mathbf{y})$ of finding simultaneously phase r at $\mathbf{x}$ and phase s at $\mathbf{y}$ is the mathematical expectation

$$P_{rs}(\mathbf{x}, \mathbf{y}) = E\left[f_r(\mathbf{x}) f_s(\mathbf{y})\right] = \int_{A} f_r(\mathbf{x}, \alpha) f_s(\mathbf{y}, \alpha) p(\alpha)\mathrm{d}\alpha. \tag{2}$$

We assume that the functions $f_r(\mathbf{x}, \alpha)$ are known [21] and there is in each of the volumes v_r, $r \in [1, n+1]$ a viscoelastic material with properties governed by the stored-energy function $W(\mathbf{F}, t)$ of third order [16] with respect to the displacement gradient

$$\frac{1}{\mu} W(\mathbf{F}, t) = \left(1 + \frac{1}{2}\alpha_1\right) I_1^2 - 2I_2 - \beta_4 I_1 T + \beta_1 I_1^3 + \beta_3 I_1 I_2 + \beta_3 I_3 + \mathrm{O}\left(|\mathbf{H}|^4\right), \tag{3}$$

where $\mathbf{F}(\mathbf{x}, t)$ is a deformation gradient and t is the time. Further, $\mathbf{H}(\mathbf{x}, t)$ denotes the gradient of the displacement vector $\mathbf{u}(\mathbf{x}, t)$ in the coordinates $\mathbf{x}$ of the reference configuration; I_k $(k = 1, 2, 3)$ are the main invariants of Lagranges's finite-strain tensor $\mathbf{E}(\mathbf{x}, t) = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1})$ and the material functions $\mu,\ \alpha_1,\ \beta_1,\ \beta_2,\ \beta_3,\ \beta_4$ stand for convolution-type integral operators of nonlinear viscoelasticity, for example

$$\mu(t) = \mu * \mathrm{d}E = \int\limits_{0^-}^{t} \mu(t-\tau)\mathrm{d}E(\tau). \tag{4}$$

Here $\mu(t)$ is a relaxation function. So the Stieltjes integral in (4), if the derivative $\dot{E}(t) = \partial E/\partial t$ exists (no jumps), can be transformed to the usual Niemann integral by setting $\mathrm{d}E(t) = \dot{E}(t)\mathrm{d}t$ [4]. The minus superscript in the lower limit of the integral indicates that the integration must begin just before $t = 0$ which is necessary when the loading starts with a jump at time $t = 0$. The functions $\mu(t),\ \alpha_1(t),\ \beta_k(t)$ $(k = 1, 2, 3, 4)$ are reduced to thermo-elastic constants in the purely thermo-elastic case $(t = 0)$. The temperature increment is denoted by T and $\mathbf{1}$ is a symmetric unit tensor of the second order [15]. So the stored energy (3) in the viscoelastic material may be expressed according to the Staverman–Schwarzl formula as follows [4]

$$W(\mathbf{F}, t) = \frac{1}{2}\int\limits_{0^-}^{t}\int\limits_{0^-}^{t} \hat{\lambda}\,(2t - t_1 - t_2)\,\mathrm{d}\mathbf{E}\,(t_1)\,\mathrm{d}\mathbf{E}\,(t_2) + \frac{1}{3}\int\limits_{0^-}^{t}\int\limits_{0^-}^{t}\int\limits_{0^-}^{t} \hat{\nu}\,(3t - t_1 - t_2 - t_3)\,\mathrm{d}\mathbf{E}\,(t_1)\ \mathrm{d}\mathbf{E}\,(t_2)\mathrm{d}\mathbf{E}\,(t_3)$$
$$- \int\limits_{0^-}^{t}\int\limits_{0^-}^{t} \hat{\beta}\,(2t - t_1 - t_2)\,\mathrm{d}\mathbf{E}\,(t_1)\,\mathrm{d}T\,(t_2).$$

After being averaged over the non-deformed representative volume v_R of the composite body B, the first asymmetric Piola stress tensor $\sigma(\mathbf{x}, t)$ and the deformation gradient $\mathbf{F}(\mathbf{x}, t)$ can be used as conjugate variables [14,16] of the nonlinear continuum theory. It follows that the state equations of the thermo-viscoelastic medium can be written as (see [4])

$$\sigma_{ka}(\mathbf{x}, \mathbf{t}) = \partial\mathbf{W}(\mathbf{F}, \mathbf{t})/\partial\mathbf{F}_{\mathbf{ka}}. \tag{5}$$

Then, for the first and second approximation of the displacement-gradient values [16] one has

$$\begin{aligned} &\sigma_{ij(1)}(\mathbf{e}) = \hat{\lambda}_{ijkl} * de_{kl(1)} + t_{ij}(T), \\ &\sigma_{ij(2)}(\mathbf{e}) = \hat{\lambda}_{ijkl} * \mathrm{d}E_{kl(2)} + [dH_{im} * \hat{\lambda}_{mjkl} * \mathrm{d}e_{kl} + de_{kl} * \hat{\nu}_{ijklmn} * \mathrm{d}e_{mn}]_{(1)}. \end{aligned} \tag{6}$$

Here

$$e_{ij} = \frac{1}{2}(H_{ij} + H_{ji}), \quad H_{ij} = F_{ij} - \delta_{ij}, \quad E_{ij(2)} = (e_{ij} + d_{ij})_{(2)},$$
$$d_{ij(2)} = \frac{1}{2}(H_{mi}H_{mj})_{(1)}, \quad \hat{\lambda}_{ijkl} = \mu(\alpha_1\delta_{ij}\delta_{kl} + 2I_{ijkl}), \quad t_{ij} = -\beta T\delta_{ij},$$
$$\hat{\nu}_{ijklmn} = \frac{1}{2}\nu_1\delta_{ij}\delta_{kl}\delta_{mn} + \nu_2(\delta_{ij}I_{klmn} + \delta_{kl}I_{ijmn} + \delta_{mn}I_{ijkl}) + 4\nu_3 I_{ijklmn},$$
$$I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad I_{ijklmn} = \frac{1}{2}(I_{ipkl}I_{jpmn} + I_{jpkl}I_{ipmn}),$$
$$\nu_1 = 2\mu\,(6\beta_1 - 3\beta_2 - 5\beta_3)\,, \quad \nu_2 = -\mu\,(\beta_2 + \beta_3)\,, \quad \nu_3 = \mu\beta_3/4, \quad \beta = \mu\beta_4.$$

The subscript in parenthesis stands for the order of approximation of the nonlinear displacement; δ_{ij} is the Kronecker delta, $\hat{\lambda}$ and $\hat{\mu}$ are convolution-type integral operators of viscoelasticity (4) or second-order Lamé elastic modules in the case of linear elastic-strain theory. The functions $\hat{\nu}_1(t)$, $\hat{\nu}_2(t)$ and $\hat{\nu}_3(t)$ are convolution-type integral operators of nonlinear viscoelasticity or Lamé third-order constants [16]. When the matrix response is nonlinear but in an elasto–visco-plastic sense rather than simply viscoelastic, it is more convenient to use the stored energy in the form (see [5,9]):

$$W(\mathbf{F},t)=\mu\left[\left(1+\frac{1}{2}\alpha_1\right)I_1^2-2I_2+\beta_1\left(I_1^2-3I_2\right)^{\frac{1+n}{2n}}-\beta_4 I_1 T\right]. \tag{7}$$

Here μ, α_1, β_1, β_4 and n are material constitutive functions or constants in the case of plastic response. Hence the stress–strain relation will be

$$\sigma_{ij}=\hat{\lambda}_{ijab}E_{ab}+\nu_3(I_1^2-3I_2)^{\frac{1-n}{2n}}K_{ijab}E_{ab}-\beta T\delta_{ij},\quad K_{ijab}=I_{ijab}-J_{ijab},\quad J_{ijab}=\frac{1}{3}\delta_{ij}\delta_{ab}. \tag{8}$$

Included in this family of materials (8) is a linear viscoelastic solid with $n=1$ and a rigid perfectly visco-plastic solid with $n=\infty$. These constitutive relations are particularly suitable for the investigation of a wide range of material behaviors [15].

3 Two-phase linear viscoelastic material

The stochastic equilibrium equations and the boundary conditions of the first linear approximation can be written in the form [3,5]

$$\begin{aligned}
&\hat{\mathbf{L}}\mathbf{u}_{(1)}(\mathbf{x},t)=\mathbf{b}_{(1)}(\mathbf{x},t),\quad \mathbf{x}\in B,\\
&\mathbf{u}_{(1)}(\mathbf{x},t)=0,\quad \mathbf{x}\in\partial B,\\
&\mathbf{b}_{(1)}(\mathbf{x},t)=-\mathbf{f}(\mathbf{x},t)\,e_{(1)}(\mathbf{x}),\quad \mathbf{f}(\mathbf{x},t)=\hat{\lambda}(\mathbf{x},t)-\hat{\lambda}^0,\\
&\sigma_{(1)}^{ij}(t)=\int_{-\infty}^{t}\hat{\lambda}_{ijkl}(t-\tau)\frac{\partial e_{kl}(\tau)}{\partial\tau}\mathrm{d}\tau-\int_{-\infty}^{t}\hat{\beta}_{ij}(t-\tau)\frac{\partial T(\tau)}{\partial\tau}\mathrm{d}\tau,\\
&\mathbf{u}_{(1)}(\mathbf{x},t)=\mathbf{u}_{(1)}^R(\mathbf{x},t)-\bar{\mathbf{u}}_{(1)}(t),
\end{aligned} \tag{9}$$

where ∂B is the boundary of a compressible composite body B, $\mathbf{u}^R(\mathbf{x},t)$ is a random displacement vector, a dash above a symbol indicating the result of a statistical averaging in the sample with random relaxation functions $\hat{\lambda}(\mathbf{x},t)$ and $\hat{\beta}(\mathbf{x},t)$. Fourier transform $\hat{\mathbf{L}}(\mathbf{k})$ of operator $\hat{\mathbf{L}}(\nabla)$ is defined in [1,5] and by others as follows

$$\left[\hat{\mathbf{L}}(\mathbf{k})\right]_{im}=\hat{\lambda}^0_{ijmn}k_jk_m,\quad \hat{\lambda}^0=\mu^0(\alpha_1^0\mathbf{1}\otimes\mathbf{1}+2\mathbf{I}). \tag{10}$$

Here $\hat{\lambda}^0$ is an elasticity tensor of the homogenous comparison body, that is, the parameters μ^0, α_1^0 are constants within the volume $v_R\in B$. The unit symmetric tensor of the fourth order is denoted by $\mathbf{I}$ and $\tau(\mathbf{x},t)$ is a stress polarization tensor [5]. The Green's function $\mathbf{u}^*(\mathbf{x})$ of the elasticity equation (10) can be defined from the following relation [3]

$$\hat{\mathbf{L}}(\nabla)\mathbf{u}^*(\mathbf{x})+\mathbf{I}\delta(\mathbf{x})=\mathbf{0}. \tag{11}$$

As $\delta(\mathbf{x})$ is the 3D Dirac function here, one has

$$u^*_{im}(\mathbf{k})=\left[\hat{\mathbf{L}}(\mathbf{k})\right]^{-1}_{im}. \tag{12}$$

Using the technique described in [18,23], one may write the solution as a convolution-type integral over the $B\cap t$ domain:

$$\mathbf{e}_{(1)}(\mathbf{x}_1,t)=\mathbf{\Gamma}(\mathbf{x}_1,\mathbf{x}_2,t,t')*\tau_{(1)}\left(\mathbf{x}_2,t'\right), \tag{13}$$

where $\mathbf{\Gamma}(\mathbf{x}_1, \mathbf{x}_2, t, t')$ is an operator with the kernel expressed through derivatives of the Green's function $\mathbf{u}^*(\mathbf{x}_1, \mathbf{x}_2, t, t')$.

Take now a two-phase isotropic material with the viscoelastic matrix being reinforced by randomly oriented spatially distributed inclusions of ellipsoidal form. The result of $\mathbf{\Gamma}(\mathbf{x}, \mathbf{y}, t, t')$ convolution with any second-order tensor function $\mathbf{b}(\mathbf{y}, t')$ may be obtained by the integral [1]

$$(\mathbf{\Gamma} * \mathbf{b})_{ij} = \int_B u^*_{a(i,j)b}(\mathbf{x}-\mathbf{y})\mathbf{b}(\mathbf{y}, t')\mathrm{d}\mathbf{y} + \oint_{\partial B} u^*_{a(i,j)}(\mathbf{x}-\mathbf{y}, t')\mathbf{b}\mathbf{n}_b\mathrm{d}\mathbf{y}. \tag{14}$$

With the boundary condition $\mathbf{b}(\mathbf{y}, t) = \mathbf{b}^0$, $\forall \mathbf{y} \in \partial B$ it can be transformed to the simpler relation [3]

$$(\mathbf{\Gamma} * \mathbf{b})_{ij} = \int_B u^*_{a(i,j)b}(\mathbf{x}-\mathbf{y}, t, t')[\mathbf{b}(\mathbf{y}, t') - \mathbf{b}^0]\mathrm{d}\mathbf{y}. \tag{15}$$

The nonlinear viscoelastic properties of the inclusions are defined by the $W_i\ (\mathbf{F}, t)$ potential, and that of the matrix are defined by the $W_m\ (\mathbf{F}, t)$ potential, i.e., $i = 1$ and $m = 2$ for a two-phase material. Averaging of (15) requires the argument coordinate $\mathbf{x}$ of the left-hand part to be placed in the v_a volume containing inclusions of the $\mathbf{n}_a$-direction, $a \in [1, n]$, which results in

$$\mathbf{e}^a\left(\mathbf{x}_1, t\right) = \mathbf{\Gamma}(\mathbf{x}_1, \mathbf{x}_2, t, t') * \left[\mathbf{f}^i \sum_{b=1}^{n} \mathbf{e}^{ba}\left(\mathbf{x}_2, \mathbf{x}_1, t'\right) p_{b|a}(\mathbf{x}_2, \mathbf{x}_1) + \mathbf{f}^m \mathbf{e}^{ma}\left(\mathbf{x}_2, \mathbf{x}_1, t'\right) p_{m|a}(\mathbf{x}_2, \mathbf{x}_1)\right]. \tag{16}$$

Here we take the following notations for the statistical-moment function $\mathbf{m}^{ba}$

$$\mathbf{m}^{ba}\left(\mathbf{x}_1, \mathbf{x}_2, t\right) = E\left[\mathbf{m}\left(\mathbf{x}_1, t\right)|\mathbf{x}_1 \in v_b, \mathbf{x}_2 \in v_a\right],$$
$$p_{b|a}(\mathbf{x}_1, \mathbf{x}_2) = p(\mathbf{x}_1 \in v_b|\mathbf{x}_2 \in v_a).$$

The probability densities of the distribution when going from the state of $\mathbf{x}_1 \in v_a$, that is from an inclusion of the $\mathbf{n}_a$-direction, to the state of $\mathbf{x}_2 \in v_b$, that is to a inclusion of the $\mathbf{n}_b$-direction, and to the state of $\mathbf{x}_2 \in v_m$, where v_m is the matrix volume, are written as follows [1]

$$\begin{aligned}
&p_{b|a}(\mathbf{x}_1, \mathbf{x}_2) = p\left(\mathbf{x}_1, \mathbf{x}_2\right)\delta_{ba} + c_b p^*\left(\mathbf{x}_1, \mathbf{x}_2\right)(1-\delta_{ba}),\\
&p\left(\mathbf{x}_1, \mathbf{x}_2\right) = c_a + c_a^*\varphi(\mathbf{x}_1, \mathbf{x}_2), \quad p_{m|a}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{c_m}\varphi^*\left(\mathbf{x}_1, \mathbf{x}_2\right),\\
&p^*\left(\mathbf{x}_1, \mathbf{x}_2\right) = 1 - p\left(\mathbf{x}_1, \mathbf{x}_2\right), \qquad \varphi^*\left(\mathbf{x}_1, \mathbf{x}_2\right) = 1 - \varphi\left(\mathbf{x}_1, \mathbf{x}_2\right),\\
&\varphi^*\left(\mathbf{x}_1, \mathbf{x}_2\right) = 1 - \varphi\left(\mathbf{x}_1, \mathbf{x}_2\right).
\end{aligned} \tag{17}$$

Here $\varphi\left(\mathbf{x}_1, \mathbf{x}_2\right)$ is a two-point correlation function of the viscoelastic field, c_a denotes volume concentration of the set X_a of inclusions of the n_a-direction and c_m is the matrix volume concentration.

Integration of Eq (9) is carried out by the technique proposed in [3,19,24], the algebraic matrix of the $\mathbf{g}$-operator obtained from the integral convolution $\mathbf{\Gamma} * \varphi$ being composed of

$$\begin{aligned}
&k_g = g_1(j_1 + r_3)/2, \quad l_g = -g_1 r_3, \quad l_g^T = l_g, \quad n_g = 2g_1(j_1 + r_3),\\
&m_g^T = g_0 j_2 + k_g, \quad m_g = g_0(1 + j_1) + 2l_g, \quad g_0 = -(2m_0)^{-r1}, \quad g_1 = -(2n_0)^{-1}, \quad r_3 = k_0\mu_0^{-1} j_3.
\end{aligned} \tag{18}$$

Here the common notations [5] are used to define the elements of the algebraic matrix $\mathbf{g}$

$$k_g = (g_{11} + g_{12})/2, \quad l_g = g_{13}, \quad l_g^T = g_{31}, \quad n_g = g_{33}, \quad m_g^T = 2g_{66}, \quad m_g = 2g_{44}. \tag{19}$$

The parameters j_1, j_2, j_3 are defined by

$$\begin{aligned}
&j_1 = \left(jw/\sqrt{r} - 1\right)/r, \quad j_2 = 1 - j_1, \quad j_3 = \left[\left(1 + 2w^2\right) j_1 - 1\right]/(2r),\\
&r = w^2 - 1, \quad j = \mathrm{arch}\,(w),
\end{aligned} \tag{20}$$

where w stands for the aspect ratio of longitude and transverse sizes of a spheroidal inclusion. Statistical strain fluctuations $\mathbf{e}^a$ of X_a-set inclusions are expressed through the mean deformation of the matrix $\mathbf{e}^m$ in the representative volume v_R of the composite body B

$$\mathbf{e}^a = c_m \mathbf{a}' \mathbf{e}^m, \quad \mathbf{a}' = \mathbf{zL}, \quad \mathbf{L}(\mathbf{x}) = \hat{\lambda}(\mathbf{x}) - \hat{\lambda}^m. \tag{21}$$

The parameter c_m denotes the volume matrix concentration; the transversally isotropic tensor $\mathbf{z}$ is given by the relation

$$\mathbf{z} = (\mathbf{g}^{-1} - \mathbf{f})^{-1}. \tag{22}$$

After averaging (21) over the set X_a of inclusions of all possible orientations, we define the statistical average strains of inclusions $\mathbf{e}^i$ and that of the matrix $\mathbf{e}^m$ through macro-strains $\mathbf{e}^0$ of the representative volume v_R

$$\mathbf{e}^i = \mathbf{A}^i \mathbf{e}^0, \quad \mathbf{e}^m = \mathbf{A}^m \mathbf{e}^0, \quad \mathbf{A}^i = \mathbf{A}^m(\mathbf{1} + \mathbf{a}), \quad \mathbf{A}^m = (\mathbf{1} + c_i \mathbf{a})^{-1}, \quad \mathbf{a} = \langle \mathbf{zL} \rangle. \tag{23}$$

Angular brackets denote the here the operation of statistical averaging over the set X of all possible orientations. Using the expressions (5), the linear relaxation functions $\lambda(t)$, $\mu(t)$ and the thermo-stress operator $\beta(t)$ for a two-component random composite material take the following form

$$\lambda^e = E(a\kappa) - 2\mu^e/3, \quad \mu^e = E(b\mu), \quad \beta^e = E(a\beta), \tag{24}$$

where

$$\kappa = (3\lambda + 2\mu)/3, \quad a_1 = a_2\left(1 + 3a_z\kappa_L\right), \quad a_2 = \left(1 + 3c_1 a_z \kappa_L\right)^{-1}, \quad b_1 = b_2\left(1 + 2b_z\mu_L\right),$$
$$b_2 = \left(1 + 2c_1 b_z \mu_L\right)^{-1}, \quad a_z = \frac{1}{3}(4k + 4l + n)_z, \quad b_z = \frac{2}{15}(k + n - l + 3m + 3p)_z.$$

Here we use the condition $c_1 = c_i$; the elements of the algebraic matrix $\mathbf{z}$ are determined from (18), (22) and κ_L, μ_L are defined by (21).

4 Multi-component compressible viscoelastic linear composites

In the case of multi-phase viscoelastic composite materials, we apply the multi-particle effective-field technique MEFM [6,11], in the refined approach of the conditional-moment method [3] and the Mori–Tanaka scheme [10,22]. Thus, consider the set of operators for the deformation field of components and the appropriate micro-values. The exact solution is assumed to exist:

$$\mathbf{e}^a = \mathbf{G}^i \mathbf{e}^m + \mathbf{e}^t = \mathbf{G}^i \mathbf{e}^m + \hat{\alpha}_t T. \tag{25}$$

Then, the tensors $\mathbf{A}^i$, $\mathbf{A}^m$, where i is the number of inclusions with the viscoelastic potential $W_i(\mathbf{F}, t)$, $i \in [1, n]$ and m is the subscript of the matrix with a viscoelastic potential $W_m(\mathbf{F}, t)$, $(m = n + 1)$, are defined through the expressions

$$\mathbf{A}^i(\mathbf{x}, t) = \mathbf{G}^i(\mathbf{x}, t)\mathbf{A}^m(t), \quad \mathbf{A}^m(t) = \mathbf{1}/E[\mathbf{G}(\mathbf{x}, t)]. \tag{26}$$

An approximate solution can be derived by replacing in the general case the unknown operator $\mathbf{G}^i$ by the approximated operator $\mathbf{T}^i$ (by the operator of the deformation concentration) for the mean deformations of inclusions denoted by i, $i \in [1, n]$, and the mean strains of the representative volume v_R, that is

$$\mathbf{e}^i(\mathbf{x}, t) = \mathbf{T}^i(\mathbf{x}, t)\mathbf{e}(\mathbf{x}, t) + \hat{\alpha}_t(\mathbf{x}, t)T. \tag{27}$$

Next, to define the tensors $\mathbf{A}^i$, $\mathbf{A}^m$, we obtain the formulas

$$\mathbf{A}^i(\mathbf{x}, t) = \mathbf{T}^i(\mathbf{x}, t)\mathbf{A}^m(t), \quad i \in [1, n]; \quad \mathbf{A}^m(t) = \mathbf{1}/E[\mathbf{T}(\mathbf{x}, t)]. \tag{28}$$

In the present investigation we define the $\mathbf{G}(\mathbf{x}, t)$ operator from the solution based on the one-point approximation of the multi-particle effective field or conditional statistical moment functions for two-phase media, [3] i.e.,

$$\mathbf{G}(\mathbf{x}, t) = \mathbf{T}(\mathbf{x}, t) = \mathbf{1} + \mathbf{a}(\mathbf{x}, t), \quad \mathbf{a}(\mathbf{x}, t) = \left\langle \mathbf{a}'(\mathbf{x}, t) \right\rangle. \tag{29}$$

Here the algebraic matrix $\mathbf{a}^i$ results from analyzing the stress–strain state in the set X_i of inclusions. Thereby, to define the tensors $\mathbf{A}^i$, $\mathbf{A}^m$ or simply $\mathbf{A}(\mathbf{x}, t)$, the following expressions are derived:

$$\mathbf{A}(\mathbf{x}, t) = \mathbf{A}^m(t)[\mathbf{1} + \mathbf{a}(\mathbf{x}, t)], \quad \mathbf{A}^m(t) = \mathbf{1}/E[\mathbf{a}(\mathbf{x}, t)]. \tag{30}$$

As in the case of incompressible materials [10], we can immediately demonstrate that the presentation of tensor $\mathbf{A}(\mathbf{x}, t)$ for a two-phase material (30) is identical to (23). In particular, this means that the accuracy of the results (30) is in agreement with the accuracy level of solutions derived by the multi-particle effective-field technique [21] or conditional statistical moment functions for multi-component composite media [3].

5 The second-order nonlinear viscoelastic solution

The equilibrium equations for statistical fluctuations of a second-order displacement $\mathbf{w}(\mathbf{x}, t) = \mathbf{u}_{(2)}(\mathbf{x}, t)$ in the representative reference volume υ_R are written in the form [16]

$$\begin{aligned}
&\hat{\mathbf{L}}(\nabla)\,\mathbf{w}(\mathbf{x}, t) = -\nabla\tau_{(2)}(\mathbf{x}, t), \quad \mathbf{x} \in B;\\
&\mathbf{w}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial B;\\
&\tau_{(2)}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, t) * d\mathbf{e}_{(2)}(\mathbf{x}, t) + \mathbf{t}(\mathbf{x}, \mathbf{e}, \mathbf{H}, t),\\
&\mathbf{t}(\mathbf{x}, \mathbf{e}, \mathbf{H}, t) = -\beta I_1 T \mathbf{1} + \frac{1}{2}\hat{\lambda}\left(\mathbf{H}^T\mathbf{H}\right)_{(1)} + \mathbf{H}_{(1)}\left(\hat{\lambda}\mathbf{e}_{(1)}\right)\hat{\nu}\,(\mathbf{e}\otimes\mathbf{e})_{(1)}.
\end{aligned} \tag{31}$$

The linear differential operator $\hat{\mathbf{L}}$ on the left side of the first equation of (31) agrees in form with the corresponding operator (9). This enables us to make use of the Green's function of the linear problem. Hence we can derive integral equation [19] defining the displacement gradient of the second-order approximation

$$\mathbf{H}_{(2)},(\mathbf{x}_1, t) = \mathbf{\Gamma}\left(\mathbf{x}_1, \mathbf{x}_2, t, t'\right) * \tau_{(2)}\left(\mathbf{x}_2, t'\right). \tag{32}$$

Statistical averaging of expression (32) is performed when the left-hand part of argument $\mathbf{x}$ is placed in the volume υ_a of an ellipsoidal inclusion with W_i properties and oriented in the $\mathbf{n}_a$-direction. Then, to define the mean deformation of inclusions oriented in this direction, we obtain the following equation

$$\mathbf{e}^a_{(2)}(\mathbf{x}_1, t) = \mathbf{\Gamma}\left(\mathbf{x}_1, \mathbf{x}_2, t, t'\right) * \sum_{b=1}^{n+1} \tau^{ba}_{(1)}\left(\mathbf{x}_2, \mathbf{x}_1, t'\right) p_{b|a}(\mathbf{x}_2, \mathbf{x}_1). \tag{33}$$

According to the proposed calculation scheme we find the solution of this equation after integrating with probability density functions of the type (17). Herewith, the nonlinear part on the right-hand side of (33) is expressed through macro-deformations $\mathbf{e}^0_{(1)}$ of the representative volume of composite that are already known in the first approximation

$$\mathbf{H}_{(1)}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}, t)\mathbf{e}^0_{(1)} + \mathbf{R}_{(1)}(\mathbf{x}, t), \quad \mathbf{x} \in B; \quad \mathbf{R} = \mathrm{skew}(\mathbf{H}) = \frac{1}{2}\left(\mathbf{H} - \mathbf{H}^T\right). \tag{34}$$

After some elementary analysis we obtain

$$\begin{aligned}
&\mathbf{e}_{(2)}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}, t)\mathbf{e}^0_{(2)} + \mathbf{e}^A(\mathbf{x}, t), \quad \mathbf{x} \in \bigcup B_i, \quad i \in [1, n];\\
&\mathbf{A}(\mathbf{x}, t) = [\mathbf{1} + \mathbf{a}(\mathbf{x}), t]\,\mathbf{A}^m(t), \quad \mathbf{a}(\mathbf{x}, t) = \langle\mathbf{zL}\rangle,\\
&\mathbf{A}^m(t) = [\mathbf{1} + E[\mathbf{a}(\mathbf{x}, t)]]^{-1}, \quad \mathbf{e}^A(\mathbf{x}, t) = [\mathbf{1} + \mathbf{a}(\mathbf{x}, t)]\,\mathbf{e}^{Am}(t) + \mathbf{b}(\mathbf{x}, t), \quad \mathbf{e}^{Am}(t) = -\mathbf{A}^m(t)\,[E(\mathbf{b})],\\
&\mathbf{b}(\mathbf{x}, t) = \langle\mathbf{zt}^L\rangle, \quad \mathbf{t}^L(\mathbf{x}, t) = \mathbf{t}\left(\mathbf{x}, \mathbf{e}_{(1)}, \mathbf{H}_{(1)}, t\right) - \mathbf{t}^m\left(\mathbf{e}_{(1)}, \mathbf{H}_{(1)}, t\right).
\end{aligned} \tag{35}$$

Here, the normalization conditions for the operators $\mathbf{A}(\mathbf{x}, t)$ and $\mathbf{e}^A(\mathbf{x}, t)$ are satisfied, i.e.,

$$E[\mathbf{A}(\mathbf{x}, t)] = \mathbf{1}, \quad E[\mathbf{e}^A(\mathbf{x}, t)] = \mathbf{0}, \quad \forall(\mathbf{x}) \in B. \tag{36}$$

By substituting the solution (35) in the averaged physical relations of second order taken from (5), we benefit from the application of the Cauchy macro-stress tensor $\mathbf{T}(t)$, the second Piola's macro-stress tensor $\mathbf{S}(t) = \mu\hat{\mathbf{S}}(t)$ and the deformation gradient $\mathbf{F}(t)$ to the multi-component compressible isotropic composite:

$$\mathbf{T}(\mathbf{F},t) = \mu J^{-1}\mathbf{F}\hat{\mathbf{S}}(\mathbf{F},t)\,\mathbf{F}^T(t), \quad J = \det(\mathbf{F}(t)),$$
$$\hat{\mathbf{S}}(\mathbf{H},t) = \alpha_1 I_1\mathbf{1} + 2\mathbf{E} - \beta_4 T + 3\beta_1 I_1^2\mathbf{1} + \beta_2 I_2\mathbf{1} + \beta_2 I_1(I_1\mathbf{1} - \mathbf{E}) + \beta_3(I_2\mathbf{1} - I_1\mathbf{E}) + \mathbf{E}^2,$$
$$\alpha_1 = \lambda/\mu, \quad \beta_1 = (\nu_1 + 6\nu_2 + 8\nu_3)/(6\mu), \quad \beta_2 = -2(\nu_2 + 2\nu_3)/\mu, \quad \beta_3 = \nu_3/\mu, \quad \beta_4 = \beta/\mu. \tag{37}$$

The overall relaxation functions of second and third order are defined by

$$\lambda^e = E(a\kappa) - 2\mu^e/3, \quad \mu^e = E(b\mu), \quad \nu^e = E(a_{im}\nu_m + b_i), \quad \beta^e = E(a\beta), \tag{38}$$

where

$$a_{11} = a^3, \quad a_{12} = 2a(a^2 - b^2), \quad a_{13} = 2l^2(a + 2b), \quad a_{22} = ab^2, \quad a_{23} = 4b^2 l, \quad a_{33} = b^3,$$
$$b_1 = 3l\,[\lambda a(a+b) + 2\mu l\,(a2b)], \quad b_2 = b^2(\lambda a + 6\mu l)/2, \quad b_3 = 3\mu b^3/4.$$

Here we have $l = (a - b)/3$; the coefficients a, b are defined by (35) and $\mu_r, \kappa_r, \nu_{1r}, \nu_{2r}, \nu_{3r}$ are the relaxation functions of second and third order of the r-component, $E(\cdot)$ denotes the statistical averaging (1). Taking into account terms of second order only, Eq. (37) takes the form

$$\hat{\mathbf{T}} = (\alpha_1 I_1 - \beta_4 T)\mathbf{1} + 2\mathbf{e} + \left(\alpha f_0 + \alpha_3 I_1^2 + \alpha_4 I_2\right)\mathbf{1} + \alpha_5 I_1\mathbf{e} + \mathbf{H}\mathbf{H}^T + \alpha_6\mathbf{e}^2 + O(|\mathbf{H}|^3),$$
$$\alpha_3 = (3\beta_1 + \beta_2 - \alpha_1)/\mu, \quad \alpha_4 = (\beta_2 + \beta_3)/\mu, \quad \alpha_5 = (2\alpha_1 - 2 - \beta_2 - \beta_3)/\mu, \quad \alpha_6 = (4 + \beta_3)/\mu. \tag{39}$$

Following the technique of [17], we have the following expressions for the tensor coefficients of the stress concentration in the elements of multi-component viscoelastic material

$$\sigma(\mathbf{x},t) = \mathbf{K}_c(\mathbf{x},\mathbf{H},t)\,\sigma^0, \quad \mathbf{K}_c(\mathbf{H},\mathbf{x},t) = \mathbf{B}(\mathbf{x},t) + \mathbf{b}(\mathbf{H},\mathbf{x},t),$$
$$\mathbf{B}(\mathbf{x},t) = \hat{\lambda}(\mathbf{x},t)\mathbf{A}(\mathbf{x},t)\hat{\mu}^e(t), \quad \hat{\mu}^e(t)\mathbf{b}(\mathbf{x},t,\sigma^0) = \hat{\lambda}(\mathbf{x},t)\mathbf{e}^A(\mathbf{x},t)/\sigma^0, \tag{40}$$

where $\hat{\mu}^e = (\hat{\lambda}^e)^{-1}$ is the overall strain compliance function. The stress concentration tensor $\mathbf{B}(\mathbf{x})$ may be written more simply as [20]

$$\mathbf{B}(\mathbf{x},t) = \mathbf{B}^m(t)[\mathbf{1} + \mathbf{b}(\mathbf{x},t)], \quad \mathbf{B}^m(t) = [\mathbf{1} + E[\mathbf{b}(\mathbf{x},t)]]^{-1}, \quad \mathbf{b}(\mathbf{x},t) = \mathbf{q}\mathbf{M}, \quad \mathbf{q} = (\mathbf{p}^{-1} - \mathbf{r})^{-1},$$
$$\mathbf{p}(t) = -\hat{\lambda}^0(+\mathbf{g}(t)\hat{\lambda}^0), \quad \mathbf{r}(\mathbf{x},t) = \hat{\mu}(\mathbf{x},t) - \hat{\mu}^0, \quad \mathbf{M}(\mathbf{x},t) = \hat{\mu}(\mathbf{x},t) - \hat{\mu}^m(t), \quad \hat{\mu}^0 = (\hat{\lambda}^0)^{-1}. \tag{41}$$

Then for a single inclusion $v_I \in B$ we will get

$$\mathbf{B}^I(\mathbf{x},t) = \mathbf{1} + \mathbf{b}(\mathbf{x},t), \quad \mathbf{b}(\mathbf{x},t) = \mathbf{q}(\mathbf{x},t)\mathbf{M}(\mathbf{x},t). \tag{42}$$

When the elasticity tensor of a comparison body is defined by $\lambda^0 = \lambda^m$ [5], the well-known Eshelby solution for the stress in a single inclusion [25] follows immediately:

$$\mathbf{B}^{IE}(t) = (\mathbf{1} - \mathbf{p}\mathbf{M})^{-1}, \quad \mathbf{p}(t) = \mathbf{1} - \hat{\lambda}^m(t)[\mathbf{1} + \mathbf{g}(t)\hat{\lambda}^m(t)]. \tag{43}$$

As a result, an effective stress-field hypothesis of MEFM [6] may be used naturally to determine the stresses in any inclusion v_I loaded by an equivalent or effective field $\tilde{\sigma}^I(\mathbf{x},t)$ in a nonlinear viscoelastic matrix.

6 Local stress field near an inclusion in nonlinear viscoelastic composites

The refined approach of the conditional-moment method (CMM) [3] with the hypothesis of a multi-particle effective-field method [6,11] is proposed here to investigate the local stress field near an inclusion in a viscoelastic random composite. We consider now a nonlinear viscoelastic composite medium [15] with stress-free strains $\mathbf{h}(\mathbf{x},t)$ consisting of a homogeneous matrix containing a homogeneous and statistically uniform random set X_i of spheroidal inclusions all having the same form, but random orientation and mechanical properties. We will be using the approach of CMM [20] with the main hypothesis of many micro-mechanical methods, according to which each inclusion is located inside a homogeneous so-called effective or equivalent stress field $\tilde{\sigma}$. It is shown, in the framework of the effective-field hypothesis [6] that, from a solution of the classical linear elastic problem with zero stress-free strains for the composite, the relations for the effective nonlinear, stored energy and average viscoelastic strains inside the components can be found.

For a single inclusion the micro-mechanical approach is based on the Green-function technique [5], as well as on the interfacial Hill operators [17,26]. As a generalization of the results [10], we consider here a certain representative meso-domain B with a characteristic function $f_B(\mathbf{x}, \alpha)$ containing a set X_I of inclusions v_I with characteristic functions $f_I(\mathbf{x}, \alpha)$, $(I = 1, 2, 3, \ldots)$. The inclusions are defined as the I-component having identical mechanical and geometrical properties. It is useful to define the equivalent field $\tilde{\sigma}$ as a stress field in which the chosen fixed inclusions are embedded. This equivalent field is a random function of all the other positions of the surrounding inhomogeneities; the average over a random realization of these inclusions is equal to the right-hand side of the integro–differential system of equations for a non-homogeneous domain [3,21,27].

More detailed considerations of the mechanical behavior of nonlinear composite materials requires an analysis of the interface between the reinforcement and the matrix [17]. The inhomogeneity of mismatch properties in the matrix is a typical situation due to both the production of the coated inclusions and thermo-visco-plastic deformation of the matrix near the inclusion. The micro-mechanical approach is based on the Green-function technique [3,5,6], as well as on the interfacial Hill operators [26]. An assumption of a homogeneous stress state in the inclusion is used. At first we consider the problem of a single inclusion inside an infinite nonlinear viscoelastic matrix. Stress and strain are related to each other via the constitutive equation (see [15])

$$\sigma(\mathbf{x}, t) = \partial W(\mathbf{x}, t)/\partial \mathbf{F}(\mathbf{x}, t) \tag{44}$$

or

$$\sigma(\mathbf{x}, t) = \hat{\lambda}(\mathbf{x}, t) * d\mathbf{e}(\mathbf{x}, t) + \mathbf{t}(\mathbf{x}, \mathbf{F}, t), \quad \mathbf{e}(\mathbf{x}, t) = \hat{\mu}(\mathbf{x}, t) * d\sigma(\mathbf{x}, t) + \mathbf{h}(\mathbf{x}, \mathbf{F}, t), \tag{45}$$

where $\hat{\lambda}(\mathbf{x}, t)$ and $\hat{\mu}(\mathbf{x}, t) = \hat{\lambda}^{-1}(\mathbf{x}, t)$ are the given phase linear relaxation and strain compliance functions, respectively, and the common notation for tensor products [16] has been employed. The thermal and nonlinear parts of the constitutive relations are represented by $\mathbf{t}(\mathbf{x}, \mathbf{F}, t)$ and $\mathbf{h}(\mathbf{x}, t) = -\hat{\lambda}(\mathbf{x}, t)\mathbf{t}(\mathbf{x}, \mathbf{F}, t)$, which are second-order tensors of the local eigenstresses and eigenstrains (transformation fields) which may arise by thermal expansion, plastic deformation, phase transformation and other changes of shape or volume of the material. We assume that the phases are perfectly bonded, so that the displacements and the traction components of the stresses are continuous across the inter-phase boundaries. We assume uniform external traction boundary conditions

$$\mathbf{T}(\mathbf{x}, t) = \sigma^0\mathbf{n}(\mathbf{x}, t), \quad \forall \mathbf{x} \in \partial B, \tag{46}$$

where $\mathbf{T}(\mathbf{x}, t)$ is the traction vector at the external boundary $\Gamma = \partial B$ of the meso-domain B, $\mathbf{n}(\mathbf{x}, t)$ is its unit outward normal vector, and σ^0 is a given constant stress tensor. Of course, the conditional-moment method presented in detail in [1] deserves to be viewed critically along with other methods. Concrete numerical results were obtained there by truncation of an infinite system of integral equations by taking into account only two-point conditional probabilities, and by neglecting fluctuations of stresses within the limits of the components. These are equivalent to adopting the assumption of homogeneous elastic fields and the consideration of homogeneous inclusions. Here we introduce a comparison body [3,5] whose mechanical properties are denoted by the upper null index. So $\hat{\lambda}^0$ will usually be taken uniform over B; as a result, the corresponding boundary-value problem is easier to solve than that for the original body with random viscoelasticity $\hat{\lambda}(\mathbf{x}, t)$. All tensors of the material properties are decomposed as

$$\mathbf{t}(\mathbf{x}, t) = \mathbf{t}^0(t) + \mathbf{t}^f(\mathbf{x}, t). \tag{47}$$

The Hill condition [26] for the elastic-energy representation holds for any compatible strain field from (47) and an equilibrium stress field σ^0 (46) not necessarily related to each other by a specific stress–strain relation. Here and in the following the upper lower-case index i indicates the components and the lower upper-case index I indicates the individual inclusion.

Let us consider some conditional statistical averages of the general integral equation (31) leading to an infinite system of integral equations (33). Concrete numerical results may be obtained for aligned or disordered homogeneous ellipsoidal inclusions under different choices of comparison media, that is, either the $\hat{\lambda}^0 = E[\hat{\lambda}(\mathbf{x})]$ or $\hat{\lambda}^0 = 1/E[\hat{\mu}(\mathbf{x})]$ estimate. Of course, there is no a priori justification for the specific choice of $\hat{\lambda}^0$, not counting the condition that the quadratic form, employed in the proof of the Hashin and Shtrikman variational principle, has a constant sign. The only justification up to the recent publication of Talbot and Willis [28] for choosing for λ^0 the Vought or Reuss estimate was the fact that specific experimental data agree with the computed curves [3]. The final general representation for effective modules was taken into account in the conditional-moment method [1]; the equivalence of the admitted assumptions leads to the conclusion that the conditional-moment method can be considered as equivalent to the one-particle approximation of MEFM. In addition, in the conditional-moment method the shape of the inclusions is taken into account via prescribed anisotropy of the conditional probability density.

For equally probable orientations of the spheroidal inclusions it is possible to obtain an isotropic function and the estimate of the effective compliance function will be invariant with respect to the shape of the inclusion. This result can be avoided easily by taking into account directly the shape of the inclusions via the tensor, as done by Willis [5] on the basis of a variational principle.

We use here the refined conditional-method approach that for randomly oriented ellipsoidal inclusions the estimate of the effective modules represented in [3,17], and [5] is equivalent. In the case of a nonlinear composite, a bound on its effective energy density does not imply a corresponding bound on its constitutive relation. Recently Talbot and Willis [28] proposed a refined method for bounding directly the constitutive relation by employing a linear comparison material. It seems a very sensible aspect of the approach proposed here that the bounds produced are closely related to bounds of Hashin–Strikman type and a comparison elasticity of Voigt and Reuss type is used. So the determination of comparison elasticity from certain experiments [1] is now established by the effective energy of a nonlinear composite evaluation.

According to Eshelby's equivalence principle [25], the perturbed strain field $\mathbf{e}'(\mathbf{x}, t)$ induced by inhomogeneities (inclusions with properties different from those of the homogeneous matrix) can be related to the specified eigenstrain by replacing the inhomogeneities with the matrix material. That is, for the domain of the r-phase inhomogeneities with the $\hat{\lambda}^r$ elasticity tensor we have

$$\hat{\lambda}^r(t)[\mathbf{e}^0 + \mathbf{e}'(\mathbf{x}, t)] = \hat{\lambda}^m(t)[\mathbf{e}^0 + \mathbf{e}'(\mathbf{x}, t) - \mathbf{e}^*(\mathbf{x}, t)], \tag{48}$$

where $\hat{\lambda}^m(t)$ is the relaxation function of the matrix and $\mathbf{e}^0$ is the uniform strain field caused by far-field loads for a homogeneous matrix material only; $\hat{\lambda}^m(t)$ and $\hat{\lambda}^r(t)$ could be isotropic or anisotropic if the eigenstrain field $\mathbf{e}^*(\mathbf{x}, t)$ is uniform in v_I. So the strain at any point within an RVE is decomposed into two parts: (a) a uniform strain $\mathbf{e}^0$ (without inhomogeneities), and (b) a perturbed strain $\mathbf{e}'(\mathbf{x}, t)$ or actual constrained strain due to the distributed "stress-free" transformation strain or eigenstrain $\mathbf{e}^*(\mathbf{x}, t)$. It is emphasized that the eigenstrain $\mathbf{e}^*(\mathbf{x}, t)$ is nonzero in the inclusion domain and zero in the matrix domain, respectively. In particular, the perturbed strain field induced by the distributed eigenstrain $\mathbf{e}^*(\mathbf{x}, t)$ can be expressed as

$$\mathbf{e}'(\mathbf{x}, t) = \int_B \mathbf{G}(\mathbf{x} - \mathbf{y})\mathbf{e}^*(\mathbf{y}, t')\mathrm{d}\mathbf{y}\,\mathrm{d}t', \tag{49}$$

where B is the domain of an RVE v_R and $\mathbf{x}, \mathbf{y} \in B$. Eshelby used a fourth-order tensor $\mathbf{S}$, which is traditionally called Eshelby's tensor, to describe the strain and stress fields in the inclusion domain. Eshelby's tensor is defined as

$$\mathbf{S}(\mathbf{x}) = \int_{B_I} \mathbf{G}(\mathbf{x} - \mathbf{y})\mathrm{d}\mathbf{y}, \tag{50}$$

in which $\mathbf{x}$ is a local point inside the inclusion domain B_I. The total strain $\mathbf{e}(\mathbf{x}, t)$ at any point $\mathbf{x} \in B_m$ in the matrix is given by a superposition of the uniform strain $\mathbf{e}^0(t)$ and the perturbed strain $\mathbf{e}'(\mathbf{x}, t)$ induced by inclusions (inhomogeneities)

$$\mathbf{e}(\mathbf{x},t)=\mathbf{e}^0(t)+\mathbf{e}'(\mathbf{x},t)=\mathbf{e}^0(t)+\int_B \mathbf{G}(\mathbf{x}-\mathbf{y})\mathbf{e}^*(\mathbf{y}(t'))\mathrm{d}\mathbf{y}. \tag{51}$$

Therefore, the volume-averaged strain tensor is given by

$$\bar{\mathbf{e}}(\mathbf{x},t)=\mathbf{e}^0(t)+\frac{1}{v_B}\int_B\int_{B_I}\mathbf{G}(\mathbf{x}-\mathbf{y})\mathbf{e}^*(\mathbf{y})\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x}=\mathbf{e}^0(t)+\frac{1}{v_B}\int_B\left[\int_B \mathbf{G}(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{x}\right]\mathbf{e}^*(\mathbf{y})\mathrm{d}\mathbf{y}. \tag{52}$$

When considering the strain and stress fields at a local point $\mathbf{x}$ outside an inclusion, we define a fourth-order tensor $\tilde{\mathbf{G}}(\mathbf{x})$, which is called the exterior-point Eshelby tensor

$$\tilde{\mathbf{G}}(\mathbf{x})=\int_{B_m}\mathbf{G}(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y},\quad \mathbf{x}\in B/B_I. \tag{53}$$

The essential assumption in the Mori–Tanaka approach states that each inclusion v_I behaves as an isolated one in the infinite matrix $\hat{\lambda}^0=\hat{\lambda}^m$ and subjected to some equivalent or effective stress field $\tilde{\sigma}(\mathbf{x}_I)=\sigma^m$ coinciding with the average stress in the matrix. This assumption allows uniquely to define the effective nonlinear elastic properties of multi-component composite materials [10,29,30]. On the other hand, this hypothesis is more restrictive than the hypothesis H1 of MEFM [6]. It gives an opportunity to use the known solution [1,5] and others for each inclusion v_I and to find the average stress in the matrix by use of a representation of the average stresses in the separate phases as the average stress in the whole composite. It makes possible to represent the statistical average of both the stresses in the matrix and the strain polarization tensor in the inclusions, as well as to an estimate of the effective properties.

Using the standard Green-function technique, we transform (31) into an integral. By doing so we obtain an estimate of the stress distribution inside the inclusion σ^I and $\sigma^\Gamma(\mathbf{s})$ in accordance with (40). Therefore, the stress-concentration tensors $\mathbf{B}(\mathbf{x},t)$, $\sigma^{BI}(\mathbf{x},t)$ in (40)–(43) are found to be

$$\mathbf{B}(\mathbf{x},t)=\text{const},\quad \sigma^{BI}(\mathbf{x})=\text{const},\quad \forall\mathbf{x}\in B_I \tag{54}$$

and

$$\mathbf{B}(\mathbf{x},t)\neq\text{const},\quad \sigma^{BK}(\mathbf{x})\neq\text{const},\quad \forall\mathbf{x}\in B_K. \tag{55}$$

After that the tensors $\mathbf{R}(\mathbf{x},t)$ and $\mathbf{e}^{RI}(\mathbf{x},t)$ are defined by (54) and the nonlinear viscoelastic properties of the homogeneous inclusions $\mathbf{M}(\mathbf{x},t)$ are evaluated by the relations (41). Hence, the nonlinear viscoelastic problem for a single inclusion is completely solved and we arrive at the estimate of the overall nonlinear relaxation functions $\hat{\lambda}^e(t)$, $\hat{\beta}^e(t)$ and $\hat{\nu}^e(t)$ from (38) and the average stresses (37) inside the components by using different tensors $\sigma^B(\mathbf{x},t)$. Some particular methods involving such tensor approaches are given in [1,5,6]. Different versions of closure assumptions in terms of conditional stress fields analogous to the hypothesis H2 of the MEFM for the effective stress fields are known [6,11]. The first-order approximations of these similar approaches and the principal difference between them is beyond the scope of direct substitution of the stress field for the equivalent field. Even for statistically homogeneous composites it may be shown that the use of the different known and useful assumptions can lead to a variation of the effective elastic modules by a factor of two or more. This fact has been confirmed by experimental data [21]. Hence, any model simplification can be evaluated precisely.

Let us now consider a simplification of the elastic solution for different particular cases of inclusions. So $\mathbf{M}(\mathbf{x},t)=\hat{\mu}(\mathbf{x},t)-\hat{\mu}^m(t)$ is the jump of the strain compliance function of any component with respect to the matrix. By this function the variation of the material properties within inclusions is taken into account. The integral-operator kernel $\mathbf{\Pi}(\mathbf{x},t)$ may be defined by the second-order Green tensor $\mathbf{u}^*$ of the Lamé equation of a homogeneous medium with an elastic modulus tensor $\hat{\lambda}^0$

$$\mathbf{\Pi}(\mathbf{x},t)=-\hat{\lambda}^0[\mathbf{1}\delta(\mathbf{x})+\mathbf{G}(\mathbf{x},t)\hat{\lambda}^0],\quad \mathbf{G}(\mathbf{x},t)=\nabla\nabla\mathbf{u}^*(\mathbf{x},t); \tag{56}$$

here $\delta(\mathbf{x})$ is the 3D Dirac delta function, $\mathbf{1}$ and $\mathbf{I}$ are the unit second- and fourth-order tensors, respectively. So we may define the strain polarization tensors $\hat{\eta}(\mathbf{x},t)$ and $\hat{\eta}^0$

$$\hat{\eta}(\mathbf{x},t)=\mathbf{R}\bar{\sigma}+\hat{\eta}^R,\quad \hat{\eta}^0=\mathbf{R}\sigma^0+\hat{\eta}^R,\quad \forall\mathbf{x}\in B_I. \tag{57}$$

Let us consider some conditional statistical averages of the general integral equation (32). In order to simplify the exact system (31), we now apply the main hypothesis of many micro-mechanical methods, the so-called effective-field hypothesis: each inclusion in domain B with measure has a spheroidal shape and is embedded in the field $\tilde{\sigma}$ which is homogeneous over the I-inclusion. The perturbation introduced by the inclusion v_I in point $\mathbf{x}$ is defined by the relation

$$\int_B f_I(\mathbf{x})\mathbf{\Pi}(\mathbf{x}-\mathbf{y})[\mathbf{M}(\mathbf{x})\sigma(\mathbf{x})+\mathbf{h}_1(\mathbf{x})]\mathrm{d}\mathbf{x} = c_I\mathbf{T}^I(\mathbf{y}-\mathbf{x})\left\langle \mathbf{M}(\mathbf{x})\sigma(\mathbf{x})+\mathbf{h}_1(\mathbf{x})\right\rangle^I, \tag{58}$$

where $\langle f\rangle^I$ is an average over the volume of the inclusion v_I and

$$c_I\mathbf{T}^I(\mathbf{y}-\mathbf{x}^I) = \int_B f_I(\mathbf{x})\mathbf{\Pi}(\mathbf{y}-\mathbf{x}^I)\mathrm{d}\mathbf{x}, \quad \mathbf{y}\notin B_I. \tag{59}$$

By analogy to [3,5,6] and in view of the linearity of every iteration, there exist constant fourth- and second-order tensors $\mathbf{B}(\mathbf{x})$ and $\sigma^B(\mathbf{x})$, respectively, such that

$$\sigma^I(\mathbf{x}) = \mathbf{B}^I(\mathbf{x})\tilde{\sigma}^I(\mathbf{x})+\sigma^{BI}(\mathbf{x}), \quad c_I[\mathbf{M}(\mathbf{x})\sigma(\mathbf{x})+\mathbf{h}_1(\mathbf{x})]^I = \mathbf{R}(\mathbf{x})\tilde{\sigma}^I(\mathbf{x})+\mathbf{e}^{RI}(\mathbf{x}), \quad \mathbf{x}\in B_I, \tag{60}$$

where the tensors $\mathbf{R}(\mathbf{x})$ and $\mathbf{e}^{RI}(\mathbf{x})$ are found by the use of the Eshelby Theorem [18]

$$\mathbf{R} = -c_i\mathbf{p}^{-1}(\mathbf{1}-\mathbf{B}), \quad \mathbf{e}^{RI}(\mathbf{x}) = c_i\mathbf{p}^{-1}\sigma^{BI}(\mathbf{x}). \tag{61}$$

The tensor $\mathbf{p}$ is associated with the Eshelby tensor $\mathbf{S}$ by

$$\mathbf{S} = \mathbf{1}+\hat{\mu}^0\mathbf{p}, \quad \mathbf{p} = \langle\mathbf{\Pi}(\mathbf{x}-\mathbf{y})\rangle^I = \text{const}, \quad \forall\mathbf{x},\mathbf{y}\in B_I. \tag{62}$$

In practice the tensors $\mathbf{B}$ and σ^{BI} are found [13,15,18] from the elastic problem of a single inclusion in the infinite matrix B_m, when

$$c_I = 0, \quad \tilde{\sigma}^I(\mathbf{x}) = \sigma^0 = \text{const}. \tag{63}$$

This problem is connected with the calculation of the inhomogeneous fourth- and second-order tensors $\mathbf{B}(\mathbf{x})$, $\sigma^{BI}(\mathbf{x})$ by either analytical or numerical methods, such that for $\mathbf{x}\in B_I$ the following holds:

$$\begin{aligned}
&\sigma(\mathbf{x}) = \mathbf{B}(\mathbf{x})\sigma^0+\sigma^{BI}(\mathbf{x}),\\
&\mathbf{B}^I = \langle\mathbf{B}(\mathbf{x})\rangle^I, \quad \sigma^{BI} = \left\langle\sigma^{BI}(\mathbf{x})\right\rangle^I,\\
&\mathbf{R} = c_I\left\langle\mathbf{M}(\mathbf{x})\mathbf{B}(\mathbf{x})\right\rangle^I, \quad \mathbf{e}^{RI} = c_I\left\langle\mathbf{M}(\mathbf{x})\sigma^{BI}(\mathbf{x})+\mathbf{h}_1(\mathbf{x})\right\rangle^I.
\end{aligned} \tag{64}$$

We consider here an analytical method for the calculation of the tensors $\mathbf{B}(\mathbf{x})$ and $\sigma^{BI}(\mathbf{x})$ for spheroidal inclusions in a sense of nonlinear field mechanics [16]. Other analytical methods for the analysis of spheroidal inclusions are mentioned in [3,9]. In the general case, estimating of the tensors $\mathbf{B}(\mathbf{x})$, $\sigma^{BI}(\mathbf{x})$ is a particular problem the analytical method involving the transformation field $\mathbf{e}^T(\mathbf{y})$ [5,6]. For the particular case of a homogeneous spheroidal domain B_I with inclusions $\mathbf{M}^I = \hat{\mu}^i-\hat{\mu}^m = \text{const}$, we have

$$\mathbf{B}^I = (\mathbf{1}+\mathbf{pM})^{-1}, \quad \sigma^{BI} = \mathbf{B}^I\mathbf{ph}_1^I, \quad \mathbf{R}^I = c_I\mathbf{MB}^I, \quad \mathbf{e}^{RI} = c_I(\mathbf{1}+\mathbf{Mp})^{-1}\mathbf{h}_1^I. \tag{65}$$

Comparing relation (5.25) with (4.10), one can see that the average nonlinear viscoelastic response (i.e., the tensors $\mathbf{B}$, σ^{BI}, $\mathbf{R}$, $\mathbf{e}^{RI}$) of any inclusion is the same as that of some homogeneous inclusion with nonlinear viscoelastic parameters which also can be expressed in terms of the tensors $\mathbf{R}$ and $\mathbf{e}^{RI}$. In the case where a single spheroidal inclusion of radius a^i is embedded in an infinite matrix, the problem may be investigated in a fairly straightforward manner:

$$\hat{\lambda}^m(3\kappa^m,2\mu^m) \equiv 3\kappa^m\mathbf{J}+2\mu^m\mathbf{K}, \quad J_{ijab} = \frac{1}{3}\delta_{ij}\delta_{ab}, \quad K_{ijab} = (\delta_{ia}\delta_{jb}+\delta_{ib}\delta_{ja})/2-J_{ijab}. \tag{66}$$

The tensors $\mathbf{t}^m = -\hat{\beta}^m T\mathbf{1} + \hat{\nu}^m(\mathbf{e}\otimes\mathbf{e})^m$ and $\mathbf{t}^i = -\hat{\beta}^i T\mathbf{1} + \hat{\nu}^i(\mathbf{e}\otimes\mathbf{e})^i$ have a special form with a physical meaning represented by the constitutive equations (5). According to Hill [26] we define the projective operators $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{E}$, $\mathbf{F}$ of the second- and fourth-order, respectively, as follows:

$$\mathbf{E} = \mathbf{1} - \mathbf{F}, \qquad \mathbf{F} = [\mathbf{f}\otimes\mathbf{f} + (\mathbf{f}\otimes\mathbf{f})^T]/2, \quad f_{ij} = \delta_{ij} - e_{ij}, \quad e_{ij} = n_i n_j. \tag{67}$$

Furthermore, the surface tensors are defined by

$$\hat{\lambda}(\mathbf{n})^{\pm} = \hat{\lambda}^{\pm}\mathbf{n}, \quad \mathbf{G}(\mathbf{n})^{\pm} = [\hat{\lambda}(\mathbf{n})^{\pm}]^{-1}, \quad \mathbf{B}^H(\mathbf{n})^{\pm} = \hat{\lambda}^{\pm}[\mathbf{1} - \mathbf{A}^H(\mathbf{n})^{\pm}\hat{\lambda}^{\pm}],$$
$$A^H(\mathbf{n})^{\pm}_{ijab} = [n_i(G(\mathbf{n})^{\pm}_{ja})n_b]_{(ij)(ab)}. \tag{68}$$

Here and below the symbols + and − relate to the different boundary sides. By testing we immediately obtain the orthogonal properties of the operators defined in [26]

$$\mathbf{ee} = \mathbf{e}, \quad \mathbf{ff} = \mathbf{m}, \quad \mathbf{em} = 0,$$
$$\mathbf{FF} = \mathbf{F}, \quad \mathbf{EE} = \mathbf{E}, \quad \mathbf{Em} = 0, \quad \mathbf{Fe} = 0, \quad \mathbf{EF} = 0. \tag{69}$$

Hence the tensors $\mathbf{A}^H(\mathbf{n})$, $\mathbf{B}^H(\mathbf{n})$ in (64) can be expressed in terms of the projective operators

$$\mathbf{A}^H(\mathbf{n}) = [\mathbf{E}\hat{\lambda}\mathbf{E}]^{-1}, \quad \mathbf{B}^H(\mathbf{n}) = [\mathbf{F}\hat{\mu}\mathbf{F}]^{-1}. \tag{70}$$

Perfect contact between two materials means

$$\mathbf{E}\sigma^+ = \mathbf{E}\sigma^-, \quad \mathbf{Fe}^+ = \mathbf{Fe}^-. \tag{71}$$

So the following relations between the stress tensors near the interface may be used here [6,17,26]

$$\sigma^- = \sigma^+ + \mathbf{B}^H(\mathbf{n})^-[(\hat{\mu}^+ - \hat{\mu}^-)\sigma^+ + (\mathbf{h}^+ - \mathbf{h}^-)], \quad \sigma^+ = \sigma^- + \mathbf{B}^H(\mathbf{n})^+[(\hat{\mu}^- - \hat{\mu}^+)\sigma - + (\mathbf{h}^- - \mathbf{h}^+)]. \tag{72}$$

Substitution of (72) in the right-hand side of (60) leads to

$$\mathbf{B}^H(\mathbf{n})^- - \mathbf{B}^H(\mathbf{n})^+ = \mathbf{B}^H(\mathbf{n})^+(\hat{\mu}^+ - \hat{\mu}^-)\mathbf{B}^H(\mathbf{n})^+. \tag{73}$$

Let a spheroidal inclusion v_I with the homogeneous compliance function $\hat{\mu}^+$ be located in an infinite homogeneous matrix with compliance function $\hat{\mu}^-$ and loaded by the homogeneous stress σ^0 on the remote boundary Γ_B. Then, according to Eshelby's theorem (with $\mathbf{h} = 0$), we have

$$\sigma^+ = \sigma^0 + \mathbf{p}^I(\hat{\mu}^+ - \hat{\mu}^-)\sigma^+ \quad \sigma^- = \sigma^0 + c_I\mathbf{T}^I(\mathbf{x}_I - \mathbf{x}^-)(\hat{\mu}^+ - \hat{\mu}^-)\sigma^+. \tag{74}$$

where the tensor $\mathbf{p}^I$ of the inclusion v_I is associated with the Eshelby tensor $\mathbf{S}$ by $\mathbf{S}^E = \mathbf{1} + \hat{\mu}^-\mathbf{p}^I$ and the tensor $\mathbf{T}^I(\mathbf{x}_I - \mathbf{x}^-)$ is defined by the relation (5.19) for the point $\mathbf{x}^- \in v_I$ near the inclusion surface $\Gamma_i = \partial B_I$. Substituting the relations (5.34) and (5.35) in (5.18), we obtain

$$c_I\mathbf{T}^I(\mathbf{x}_I - \mathbf{x}^-) = \mathbf{B}^H(\mathbf{n})^- + \mathbf{p}^I \tag{75}$$

In particular for an isotropic medium with the viscoelastic relaxation function $\hat{\lambda}(t)$, the inversion of the matrix $\hat{\lambda}(\mathbf{n})$ may be simplified

$$\hat{\lambda}(\mathbf{n})_{ij} = \mu\delta_{ij} + (\kappa + \mu/3)e_{ij}, \quad G_{ij} = \frac{1}{\mu}\left(\delta_{ij} - \frac{2\kappa + \mu}{3\kappa + 4\mu}e_{ij}\right),$$
$$\mathbf{A}^H(\mathbf{n})_{ijab} = \frac{1}{m}\left(E_{ijab} - \frac{3\kappa - 2\mu}{3\kappa + 4\mu}e_{ij}e_{ab}\right), \quad m = 2\mu, \quad \mathbf{B}^H(\mathbf{n})_{ijab} = m\left(F_{ijab} + \frac{3\kappa - 2\mu}{3\kappa + 4\mu}f_{ij}f_{ab}\right). \tag{76}$$

The matrix stresses in the immediate vicinity of the inclusions v_I denoted by $\sigma_I^-(\mathbf{n})$, are given by the formula

$$\sigma_I^-(\mathbf{n}) = \sigma_I^+(\mathbf{n}) + \mathbf{B}^H(\mathbf{n})[\mathbf{M}(\mathbf{x})\sigma_I^+ + \mathbf{h}_I(\mathbf{x})]. \tag{77}$$

where $\sigma_I^-(\mathbf{n})$ and $\sigma_I^+(\mathbf{x})$ are the limiting stresses outside and inside, respectively, near the inclusion boundary $\Gamma_I = \partial B_I$

$$\sigma_I^-(\mathbf{n}) = \lim \sigma(\mathbf{y}), \quad \mathbf{y} \to \mathbf{x}, \quad \mathbf{y} \in v_{m},$$
$$\sigma_I^+(\mathbf{x}) = \lim \sigma(\mathbf{z}), \quad \mathbf{z} \to \mathbf{x}, \quad \mathbf{z} \in v_I, \quad \mathbf{x} \in \Gamma_I. \tag{78}$$

Here $\mathbf{n}$ is the unit outward normal vector on Γ_I. The relation (78) is correct for any shape of the inclusion v_I. The tensor $\mathbf{B}^H(\mathbf{n})$ depends only on the viscoelastic properties of the matrix material $\hat{\lambda}^m(t)$ and on the direction of the normal $\mathbf{n}$. The expression for $\mathbf{B}^H(\mathbf{n})$ is $\mathbf{H}^B$ presented as follows

$$\mathbf{B}^H(\mathbf{n})_{ijab} = m\left(F_{ijab} + \frac{3\kappa - 2\mu}{3\kappa + 4\mu} f_{ij} f_{ab}\right). \tag{79}$$

By rearranging the latter equation into an integral equation and transforming it by a method developed earlier [31,32], we obtain

$$\sigma(\mathbf{x}) - \mathbf{\Pi}(\mathbf{x}-\mathbf{y}) * \hat{\eta} = \sigma^0, \quad \mathbf{\Pi} = -\hat{\lambda}^0(\mathbf{I} + \mathbf{g}\hat{\lambda}^0), \quad \hat{\eta} = \mathbf{h} + \hat{\gamma}, \quad \hat{\gamma} = \mathbf{y}\sigma, \quad \mathbf{y} = \hat{\mu} - \hat{\mu}^0. \tag{80}$$

The jump of the strain compliance function $\mathbf{M}(\mathbf{x}, t)$ of the r-component with respect to the matrix (m-component) is

$$\mathbf{M}(\mathbf{x}, t) = \hat{\mu}(\mathbf{x}, t) - \hat{\mu}^m(t) \tag{81}$$

The integral-operator kernel is defined by the Green tensor $\mathbf{G}(\mathbf{x}, t)$ of the Lamé equation of a homogeneous comparison medium with viscoelasticity $\hat{\lambda}^0(t)$.

Equation (81) means that the average stress $\langle\sigma\rangle = \sigma^0$ is precisely determined and that the average strain $\langle\mathbf{e}\rangle = \mathbf{e}^0$ can be measured in terms of the boundary displacements. As a special case of spheroidal inclusions let us assume that $a^1 = a^2$, where the spheroid aspect ratio is defined as $w = a_3/a_1$. Following [3,33], if all inclusions and the matrix are viscoelastic, then for the particular case of the homogeneous spheroidal domain B_I with $\mathbf{M} = \hat{\mu}^i - \hat{\mu}^m = \text{const}$ we have

$$\mathbf{B} = (\mathbf{1} - \mathbf{pM})^{-1}. \tag{82}$$

The tensor $\mathbf{p}$ is associated with Eshelby's tensor $\mathbf{S}$ [25] by

$$\mathbf{S} = \mathbf{1} - \mathbf{pM}, \qquad \mathbf{p} = -\hat{\lambda}^0(\mathbf{1} + \mathbf{g}\hat{\lambda}^0). \tag{83}$$

The Eqs. (72) and (83) allow one to estimate the ensemble average of the matrix stresses in the vicinity of the inclusions near a boundary point $\mathbf{x} \in \Gamma_{BI}$ of inclusion B_I.

7 Examples of numerical implementation

One of the objectives here is to study the nonlinear material response of polymer matrix-based composites. As an example we select a glass-boron/epoxy material system. Epoxy is used as a bonding agent. The fiber is assumed to remain elastic during deformation so that the inelastic effects are limited to the matrix phase. We will use a Rabotnov-type kernel [3]

$$R(t) = \tau^{m-1}\sum_{n=0}^{\infty}(-1)^n (t/\tau)^{z-1}\Gamma(z), \quad z = m(1+n), \quad \tau = b^{-1/m}, \tag{84}$$

where m, b are viscoelastic parameters, $\Gamma(z)$ is the gamma function and $m = 1+\alpha$, $b = -\beta$ in Rabotnov's original version with operator $E_\alpha(-\beta)$.

Although some experimental observations advocate a pressure-dependent behavior of such materials, the present approach assumes negligible volume deformation during viscoelastic deformation [1]. Shear-deformation can be described by the integral operator

$$\tilde{\mu}_m = \mu_0\left[1 - \xi\hat{R}(m, b)\right], \quad \hat{R}(m, b) * e(t) = \int_0^t R(m, b, t-s)\, e(s)\, \mathrm{d}s. \tag{85}$$

or

$$\hat{R}(m, b) * e_0 = \tau^m\left\{1 - \sum_{n=0}^{\infty}(-1)^n x^n / \Gamma[m(1+n)]\right\} e_0,$$

$$x = (t/\tau)^m, \tag{86}$$

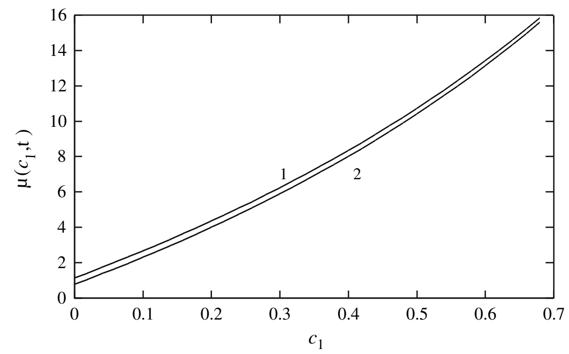

Fig. 1 Shear relaxation function $\mu(c_1, t)$

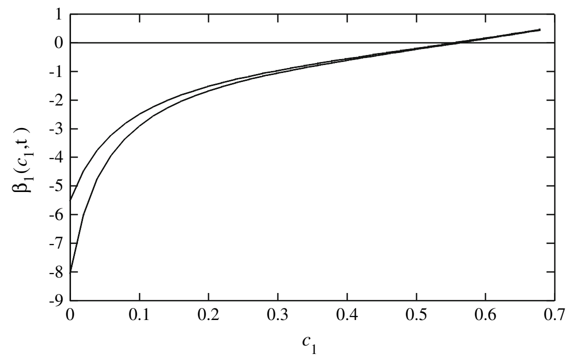

Fig. 2 Second-order relaxation function $\beta_1(c_1, t)$

Fig. 3 Second-order relaxation function $\beta_3(c_1, t)$

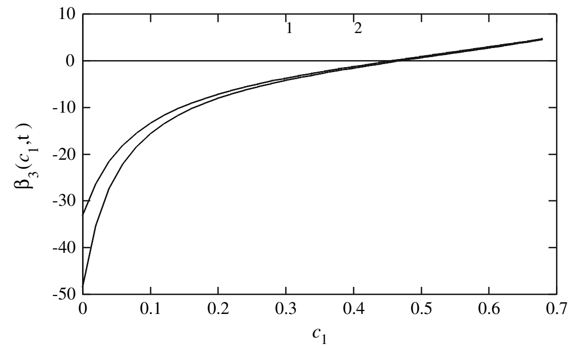

Table 1 Material constants

Material	μ(GPa)	α_1	β_1	β_2	β_3	$\alpha \cdot 10^6$ (K^{-1})
Glass	29.2	0.66	13.29	1.03	14.38	4.6
Boron	172.4	2.45	−37.6	9.40	−9.05	8.3
Epoxy	1.14	3.24	−33.12	9.40	−33.16	65.0

Table 2 Epoxy viscoelastic parameters

Material	ξ (h)$^{-m}$	m	b (h)$^{-m}$
Epoxy	0.0564	0.5	0.1764

if $e_0 = \text{const}$. The results shown in Fig. 1 illustrate the dependence of the shear-effective relaxation function $\mu(t)$ on the volume concentration of glass c_1. Curve 1 corresponds to a pure-elastic solution; curve 2 follows as a result of a viscoelastic analysis for $t = 10\tau$ where, according to (86), τ is, relaxation time of Rabotnov's kernel (84)

Further, Fig. 2 shows the variation of the second-order relaxation function $\beta_1(t)$ when the volume glass concentration c_1 increases. Figure 3 shows the dependence of the second-order relaxation function $\beta_3(t)$ on the volume concentration of glass c_1. Curve 1 corresponds to a purely elastic solution; curve 2 is determined as a result of a viscoelastic analysis for $t = 10\tau$. The calculation has been carried out with thermo-elastic parameters of the constituents as given in Table 1 and the viscoelastic properties of epoxy are presented in Table 2.

8 Conclusions

A nonlinear thermo-viscoelastic problem has been investigated in second-order approximation theory when gradient deformation terms higher than second order are neglected. A convex potential function in the thermo-elasticity and a time functional for the viscoelastic case are used to construct overall constitutive relations. Hill's technique of surface operators as developed in [26] and by others is used to determine the viscoelastic stress concentration near inclusions in polymer matrix composites (PMC) for nonlinear matrix creep.

Initiation of failure in a PMC specimen [12] can be related to the nonlinear viscoelastic stress field near inclusions. Nevertheless, more often one examines the distribution of extreme values for the stress in or near the elastic inclusions. The early analytical work on the damage of the PMC used linear elastic mechanics, so it has been less successful in applications than that applied to metals. Hence new approaches have been created lately in composite micro-mechanics [3] to investigate a stress distribution caused by external loading and interaction of structural inhomogeneities. Additional fundamental difficulties appear in the analysis of micro–macro problems when micro-inclusions and their spacing have a length-scale that is a few orders of magnitude smaller than the length-scales of the macroscopic problem. So it is very important that the displacements outside the inclusion and the stresses on the inclusion surface become known, once the inclusion strains have been computed.

It is well known that the problem involving geometrically nonlinear materials has hardly been investigated. For deriving the second-order elasticity equations, one uses the method of successive approximation [16] with a power-series expansion of the displacements, stresses, and their gradients in a certain small parameter. The case of a finite concentration of ellipsoidal elastic inclusions was analyzed in [10,18,20] by the method of conditional moments at each step of a successive approximation. The average strains in the components estimated at the first step were used at the second step or, in other words, one used linearization. Other methods for analyzing nonlinear properties are based on the incremental method or the tangent modulus concept [9]. Other remarkable achievements in the theory of nonlinear composites of random structures are related to generalizations of Hashin and Shtrikman variational principles that are applicable to nonlinear materials as proposed in [5,28]. In contrast to the extensive work on the linear elastic behavior of composites, a study of their mechanical behavior beyond the linear elastic regime, in particular, the viscoelastic one, is very limited.

Since methods of averaged strains only allow estimating the averaged stresses in the components, their use for linearizing functions describing nonlinear effects, e.g. strength predictions, creep and rupture, might lead to crude estimates because of the significant inhomogeneity of the stress fields in the individual phases, especially in the matrix near inclusions. So we have obtained here expressions for the relaxation functions and stress-concentration parameters averaged over the viscoelastic matrix and the elastic inclusions.

Within the proposed method one constructs a hierarchy of statistical-moment equations for the conditional averages of the stresses in the matrix and inclusions. The influence of the shape and the orientation of the inclusions on the viscoelastic fields, as well as stress-concentration factors near inclusions are estimated.

Some examples show the importance of the mutual influence of the thermo-elastic and viscous properties of the constituents on stress redistribution near inclusions in multi-component PCM. A more detailed numerical analysis will be presented in a next paper.

The most important extension of this work concerns using it to determine the second moment of the viscoelastic stress playing a fundamental role in a wide class of nonlinear viscoelastic problems, damage initiation and evolution, etc. The case of a viscoelastic inclusion *and* a matrix will be considered next. The developed method has a variety of applications in the mechanics of composites. Special attention may be given to the problem of the continuum estimate of the effective thermo-viscoelastic properties of nanocomposites.

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